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On Schauder's Fixed Point Theorem and Forced Second-Order Nonlinear Oscillations*

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Let $e(t)$ be continuous, periodic with period $T > 0$, and have mean zero. In this note we will show that for any number c the differential equation

$$\ddot{x} + c\dot{x} + g(x) = e(t) \tag{1}$$

has a T-periodic solution provided that g is continuous, $xg(x) \geqslant 0$ for $|x|$ sufficiently large, and $g(x)/x \to 0$ as $|x| \to \infty$. Our proof makes use of what appears to be a new method of applying the Schauder fixed point theorem to establish the existence of periodic solutions of nonlinear differential equations. In a future paper we hope to be able to formulate this method in a general setting and thereby establish the existence of periodic solutions of more general nonlinear differential equations.

For brevity we introduce some notation. P will denote the set of real-valued continuous functions with period T. $\underset{\sim}{Q}$ will denote the set of $f \in P$ with

$$\int_0^T f(s)\, ds = 0. \qquad \text{If} \quad f \in P, \quad \|f\| \quad \text{will denote} \quad \max|f(t)|.$$

For ease in proving the first part of our main result, we state three easily established lemmas.

LEMMA 1. *If* $f \in \underset{\sim}{Q}$ *and* $I(f)(t) = \int_0^t f(s)\, ds$, *then* $I(f) \in P$ *and* $\|I(f)\| \leqslant T/2 \|f\|$.

LEMMA 2. *If* $F \in P$ *and* $G(F)(t) = [e^T - 1]^{-1} \int_t^{T+t} e^{-(t-s)} F(s)\, ds$, *then* $G(F) \in P$, $\|G(F)\| \leqslant \|F\|$, *and* $G(F)$ *is a solution of the differential equation*

$$\dot{x} + x = F(t).$$

From Lemmas 1 and 2 we obtain:

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LEMMA 3. *If* $f \in Q$ *and* $H(f) \equiv G(I(f))$, *then* $H(f) \in P$, $\| H(f)\| \leqslant \| I(f)\| \leqslant (T/2)\| f \|$.

THEOREM. *Let* $e \in Q$. *If* g *is continuous, if*

$$g(x)/x \to 0 \quad as \quad |x| \to \infty, \tag{2}$$

and if there exists a number b *such that*

$$xg(x) \geqslant 0 \quad for \quad |x| \geqslant b, \tag{3}$$

then for any number c *the differential equation*

$$\ddot{x} + c\dot{x} + g(x) = e(t)$$

has at least one T*-periodic solution.*

PROOF: *Case I.* $c \neq 0$.

In this case it suffices to consider $c = 1$ for under the change of independent variable $s = ct$, the equation takes the form $x'' + x' + h(x) = E(s)$, $' = (d/ds)$ where $h(x) = g(x)/c^2$, $E(s) = e(s/c)/c^2$, so that $h(x)\, x \geqslant 0$ for $|x| \geqslant b$, $h(x)/x \to 0$ as $|x| \to \infty$, and $E(s + S) \equiv E(s)$, $\int_0^S E(v)\, dv = 0$, if $S = |c|\, T$.

We note that the condition (2) implies that for any $\epsilon > 0$ there exists a number $L(\epsilon)$ such that

$$|g(x)| \leqslant \epsilon D \qquad \text{if} \qquad D \geqslant L(\epsilon) \qquad \text{and} \qquad |x| \leqslant D. \tag{4}$$

Indeed, if $r(\epsilon)$ is such that $|g(x)| \leqslant \epsilon\, |x|$ for $|x| \geqslant r(\epsilon)$, if

$$M = \max\{|g(x)| \mid |x| \leqslant r(\epsilon)\}$$

and $L(\epsilon) \equiv \max(r(\epsilon), M/\epsilon)$, $L(\epsilon)$ satisfies (4).

If $\theta \in P$, let us define

$$\hat{g}(\theta)(t) = g(\theta(t)) - N(\theta); \qquad N(\theta) = \frac{1}{T}\int_0^T g(\theta(s))\, ds. \tag{5}$$

Clearly, for all $\theta \in P$, $\hat{g}(\theta) \in Q$ and

$$\| \hat{g}(\theta)\| \leqslant 2\epsilon D \qquad \text{if} \qquad D \geqslant L(\epsilon) \qquad \text{and} \qquad \| \theta \| \leqslant D. \tag{6}$$

Let R denote the real numbers and let $B = P \times R$. If (θ, a), (θ_1, a_1), $(\theta_2, a_2) \in B$, $x_1, x_2 \in R$, let us define

$$|(\theta, a)| = \| \theta \| + |a|,$$

$$x_1(\theta_1, a_1) + x_2(\theta_2, a_2) = (x_1\theta_1 + x_2\theta_2,\, x_1 a_1 + x_2 a_2).$$

With these definitions $(B, |\ |)$ is a complete normed linear space. If for each $(\theta, a) \in B$ we define $A[(\theta, a)] = (\theta^*, a^*)$ where

$$\begin{aligned} \theta^* &= a + H[e - \hat{g}(\theta)] \\ a^* &= a - N(\theta^*) \end{aligned} \tag{7}$$

then, by Lemma 3, A is a continuous mapping of B into B.

Let

$$0 < \delta < \min\{1/3,\ 1/3T\},$$

$$D = \max\{b/(1-3\delta),\ (b + (3/2)\,T\,\|e\|)/(1-3T\delta),\ L(\delta)\} \tag{8}$$

where $L(\delta)$ is as in (4), and let

$$m = \max\{\delta D,\ (T/2)\|e\| + T\delta D\} \tag{9}$$

so that

$$b + 3m \leqslant D. \tag{10}$$

Let

$$K = \{(\theta, a) \in B \mid \|\theta\| \leqslant D,\ |a| \leqslant b + 2m\} \tag{11}$$

so that K *is a closed and convex subset of* B. We assert:

(i) $A(K) \subset K$,

(ii) $A(K)$ *is conditionally compact* ($A(K)$ *closure compact*).

To prove (i) consider $(\theta, a) \in K$, from (6)–(11) and Lemma 3,

$$\begin{aligned}\|\theta^*\| &\leqslant |a| + \|H[e - \hat{g}(\theta)]\| \\ &\leqslant (b + 2m) + (T/2)\|e - \hat{g}(\theta)\| \\ &\leqslant (b + 2m) + T/2(\|e\| + 2\delta D) \leqslant b + 3m \leqslant D.\end{aligned} \tag{12}$$

If $-(b + m) \leqslant a \leqslant b + m$, then since

$$D \geqslant L(\delta) \qquad \text{and} \qquad \|\theta^*\| \leqslant D,$$

$$\|N(\theta^*)\| = \left\| \frac{1}{T}\int_0^T g(\theta^*(s))\,ds \right\| \leqslant \delta D \leqslant m,$$

so that $-(b + 2m) \leqslant a - N(\theta^*) \leqslant b + 2m$, and so

$$a \in [-(b + m), (b + m)] \text{ implies } a^* \in [-(b + 2m), b + 2m]. \tag{13}$$

By (6), (7), and (9),

$$\|\theta^* - a\| = \|H[e - \hat{g}(\theta)]\| \leqslant (T/2)(\|e\| + 2\delta D) \leqslant m,$$

so that $a \geqslant b + m$ implies $\theta^*(t) \geqslant b$ and $a \leqslant -(b + m)$ implies $\theta^*(t) \leqslant -b$ for all t. Hence, by (3) $a \geqslant b + m$ implies $g(\theta^*(t)) \geqslant 0$, and $a \leqslant -(b + m)$ implies $g(\theta^*(t)) \leqslant 0$ for all t. Hence, by (3) and (5) $(b + m) \leqslant a \leqslant (b + 2m)$ implies $b \leqslant a - N(\theta^*) \leqslant a \leqslant b + 2m$ and $-(b + 2m) \leqslant a \leqslant -(b + m)$ implies $-(b + 2m) \leqslant a \leqslant a - N(\theta^*) \leqslant -b$. Hence,

$$\begin{aligned} a &\in [b + m\ b + 2m] &&\text{implies} \quad a^* \in [b, b + 2m], \\ a &\in [-(b + 2m), -(b + m)] &&\text{implies} \quad a^* \in [-(b + 2m), -b]. \end{aligned} \tag{14}$$

Assertion (i) follows from (11)–(14).

To prove assertion (ii) we must show that if $\{(\theta_n^*, a_n^*)\} = \{A(\theta_n, a_n)\}$ is a sequence in $A(K)$, then there exists a subsequence $\{(\theta_{n_k}^*, a_{n_k}^*)\}$ of $\{(\theta_n^*, a_n^*)\}$ and an element $(\bar{\theta}, \bar{a}) \in B$ such that

$$\lim_{n_k \to \infty} |(\theta_{n_k}^*, a_{n_k}^*) - (\bar{\theta}, \bar{a})| = 0.$$

Suppose then $\{(\theta_n^*, a_n^*)\} = \{A(\theta_n, a_n)\}$ is such a sequence. We consider the functions $v_n = H[e - \hat{g}(\theta_n)] = G[I(e - \hat{g}(\theta_n))]$. By (9) and Lemmas 1–3,

$$\| v_n \| \leqslant (T/2) \| e - \hat{g}(\theta_n) \| \leqslant (T/2)(\| e \| + 2\delta D) \leqslant m$$

and

$$\left\| \frac{dv_n}{dt} \right\| = \| -v_n + I(e - \hat{g}(\theta_n)) \| \leqslant \| v_n \| + (T/2) \| e - \hat{g}(\theta_n) \| \leqslant 2m.$$

Hence, the sequence $\{v_n\}$ is *equicontinuous and uniformly* bounded, and thus, since $\{v_n\} \subset P$, by Ascoli's Lemma, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a $w \in P$ such that

$$\lim_{n_k \to \infty} \| v_{n_k} - w \| = 0.$$

From the condition $a_{n_k} \in [-(b + 2m]$, we may assume by again taking subsequences that

$$\lim_{n_k \to \infty} a_{n_k} = \alpha$$

exists and so by (7),

$$\lim_{n_k \to \infty} \theta_{n_k}^*(t) = \lim_{n_k \to \infty} (a_{n_k} + v_{n_k}(t)) = \alpha + w = \bar{\theta}(t)$$

uniformly in t. Obviously,

$$\lim_{n_k \to \infty} a_{n_k}^* = \lim_{n_k \to \infty} (a_{n_k} - N(\theta_{n_k}^*)) = \alpha - N(\bar{\theta}) \equiv \bar{a};$$

and hence,

$$\lim_{n_k \to \infty} |(\theta_{n_k}^*, a_{n_k}^*) - (\bar{\theta}, \bar{a})| = \lim_{n_k \to \infty} (\| \theta_{n_k}^* - \bar{\theta}_{n_k} \| + | a_{n_k}^* - \bar{a} |) = 0,$$

which proves assertion (ii).

To conclude the proof of Theorem 1, Case I, we note that (i), (ii), the fact that K is closed and convex, and the Schauder Fixed Point Theorem as given in [1, p. 131] imply the existence of $a(\phi, \hat{a}) \in K$ such that $(\phi, \hat{a}) = A[(\phi, \hat{a})] = (\phi^*, \hat{a}^*)$.

Therefore by (7),

$$N(\phi) = 0, \qquad \hat{g}(\phi) = g(\phi),$$

$$\phi = \hat{a} + H[e - g(\phi)],$$

and so by Lemma 3, $\phi \in P$ and

$$\ddot{\phi} + \dot{\phi} + g(\phi) = e(t).$$

Case II. $c = 0$.

To take care of this case we need a substitute for Lemma 3.

If $f \in P$, we define $f^{\#}(t) = f(t) - (1/T) \int_0^T f(s)\,ds$. It follows immediately that

$$f^{\#} \in Q, \|f^{\#}\| \leq 2\|f\|. \tag{15}$$

LEMMA 4. *If $f \in Q$ and $S(f) \equiv I(I(f^{\#}))$, then*

$$S(f) \in P \quad \|S(f)\| \leq T^2/2\|f\|$$

and $S(f)$ is a solution of

$$\ddot{x} = f(t). \tag{16}$$

PROOF. By Lemma 1 and (15) $I(f) \in P$ and $I(f)^{\#} \in Q$, another application of Lemma 1 implies that $S(f) = I(I(f)^{\#}) \in P$. Moreover, by Lemma 1 and (15),

$$\|I(I(f)^{\#})\| \leq (T/2)\|I(f)^{\#}\| \leq T\|I(f)\| \leq T^2/2\|f\|.$$

If $x = I(I(f)^{\#})(t)$, then

$$\dot{x} = I(f)^{\#}(t) = \int_0^t f(s)\,ds - \frac{1}{T}\int_0^T \left(\int_0^s f(u)\,du\right) ds.$$

Another differentiation gives (16).

CONCLUSION OF PROOF. For $(\theta, a) \in P \times R$ define $E(\theta, a) = (\theta^*, a^*)$ where

$$\theta^* = a + S(e - \hat{g}(\theta)),$$

$$a^* = a - N(\theta^*).$$

By mimicking the proof for Case I we show that E has a fixed point $(u, \hat{b})$ and u is T-periodic solution of

$$\ddot{x} + g(x) = e(t).$$

REFERENCE

1. J. CRONIN. "Fixed Points and Topological Degree in Nonlinear Analysis," Math. Survey 11, American Mathematical Society, Providence, 1964.

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